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The Asymptotic Expansion for Approximation Operators of Favard–Szász Type

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Abstract

We derive the complete asymptotic expansion for Favard–Szász type operators introduced by Jakimovski and Leviatan. Furthermore, we treat simultaneous approximation.

1 Introduction

In 1969, A. Jakimovski and D. Leviatan [16] introduced a Favard–Szász type operator, by means of Appell polynomials.

Throughout this paper let $g(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ be a function analytic in the disk $|z| < R$ $(R > 1)$ with $g(1) \neq 0$. The Appell polynomials $p_\nu(x)$ $(\nu = 0, 1, \ldots)$ are defined by the equation

$$g(z)\ e^{xz} = \sum_{\nu=0}^{\infty} p_\nu(x)\, z^\nu. \tag{1}$$

Let E be the class of all functions of exponential type which satisfy the property $|f(t)| \leq ce^{At}$ $(t \geq 0)$ for some finite constants c, $A > 0$.

The Jakimovski–Leviatan operators P_n $(n = 1, 2, \ldots)$ associate to each function $f \in E$

$$P_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{\nu=0}^{\infty} p_\nu(nx)\ f\left(\frac{\nu}{n}\right) \qquad (x \geq 0). \tag{2}$$

Note that the operators P_n are well–defined, for all sufficiently great n, since the infinite sum in (2) is convergent if $n > A/\log R$.

B. Wood [22] proved that the operator P_n is positive in $[0, \infty)$ if and only if $a_\nu / g(1) \geq 0$ $(\nu = 0, 1, \ldots)$. Throughout this paper we will assume that the operators P_n are positive.

In the special case $g(z) \equiv 1$ we get back the classical operators S_n of Szász–Mirakjan

$$S_n(f;x) = e^{-nx} \sum_{\nu=0}^{\infty} \frac{(nx)^\nu}{\nu!}\ f\left(\frac{\nu}{n}\right) \qquad (x \geq 0).$$

In [16] Jakimovski and Leviatan obtained several approximation properties of the operators (2). They proved that, for all $f \in C[0, \infty) \cap E$,

$$\lim_{n \to \infty} P_n(f;x) = f(x),$$

the convergence being uniform in each compact subset of $[0, \infty)$.

In a recent paper [10] A. Ciupa studied the rate of convergence of the operators (2).

The purpose of this paper is to derive the complete asymptotic expansion

$$P_n(f;x) \sim f(x) + \sum_{k=1}^{\infty} c_k(f;x)\, n^{-k} \qquad (n \to \infty), \tag{3}$$

provided f admits derivatives of sufficiently high order at $x \geq 0$. The coefficients $c_k(f;x)$ depend on the function g, but are independent of n. Formula

(3) means that, for all $m = 1, 2, \ldots$, there holds

$$P_n(f;x) = f(x) + \sum_{k=1}^{m} c_k(f;x)\, n^{-k} + o(n^{-m}) \qquad (n \to \infty).$$

Moreover, we study simultaneous approximation. It turns out that, for $\ell = 0, 1, \ldots$, there holds

$$\left(\frac{d}{dx}\right)^{\ell} P_n(f;x) \sim f^{(\ell)}(x) + \sum_{k=1}^{\infty} \left(\frac{d}{dx}\right)^{\ell} c_k(f;x)\, n^{-k} \qquad (n \to \infty).$$

In the special case $g(z) \equiv 1$ we obtain the complete asymptotic expansion of the Szász–Mirakjan operators and their derivatives.

Jakimovski and Leviatan [16, Theorem 5] defined also, for locally integrable functions f, a Kantorovich variant

$$K_n(f;x) = \frac{ne^{-nx}}{g(1)} \sum_{\nu=0}^{\infty} p_\nu(nx) \int_{\frac{\nu}{n}}^{\frac{\nu+1}{n}} f(t)\ dt \qquad (x \geq 0), \tag{4}$$

of their operators. We present the complete asymptotic expansion for these operators, too.

We mention that analogous results for the Bernstein–Kantorovich operators, the Meyer–König and Zeller operators and the operators of Butzer, Bleimann and Hahn can be found in [1, 2, 3, 4, 5]. Similar results on a certain positive linear operator can be found in [7, 15].

2 Main Results

Let $q \in \mathbb{N}$. For a fixed $x \in [0, \infty)$, let $K^{[q]}(x)$ be the class of all functions $f : E \to \mathbb{R}$ such that f admits a derivative of order q at x.

Theorem 1 (Complete asymptotic expansion for the operators P_n). *Let $q \in \mathbb{N}$, $x \geq 0$, and $f \in K^{[2q]}(x)$. The Jakimovski–Leviatan operators satisfy the asymptotic relation*

$$P_n(f;x) = f(x) + \sum_{k=1}^{q} c_k(f;x)\, n^{-k} + \mathrm{o}\left(n^{-q}\right) \qquad (n \to \infty), \tag{5}$$

where the coefficients $c_k(f;x)$ are given by

$$c_k(f;x) = \sum_{s=0}^{k} a(k,s)\, \frac{x^s}{s!} f^{(k+s)}(x) \tag{6}$$

with

$$a(k,s) = \frac{1}{k!} \sum_{m=0}^{k} \binom{k}{m} \frac{g^{(m)}(1)}{g(1)} \sum_{r=0}^{s} (-1)^{s-r} \frac{\binom{s}{r}}{\binom{r+k}{r+m}} \sigma(r+k, r+m)\ . \tag{7}$$

The quantities $\sigma(n,k)$ denote the *Stirling numbers* of the second kind. Recall that they are defined by the relations

$$x^n = \sum_{k=0}^{\infty} \sigma(n,k)\, x^{\underline{k}} \qquad (n = 0, 1, \ldots), \tag{8}$$

where, for $x \in \mathbb{R}$, $x^{\underline{n}}$ denotes the falling factorial defined by $x^{\underline{0}} = 1$ and

$$x^{\underline{n}} = x(x-1)\cdots(x-n+1) \qquad (n = 1, 2, \ldots).$$

Note that $\sigma(n,k) = 0$ if $k > n$.

Concerning simultaneous approximation we have the following result.

Theorem 2 (Complete asymptotic expansion for the derivatives of the operators P_n). *Let $q \in \mathbb{N}$, $x \geq 0$, $\ell \in \mathbb{N}_0$ and $f \in K^{[2q+2\ell]}(x)$. The derivatives of the Jakimovski–Leviatan operators satisfy the asymptotic relation*

$$P_n^{(\ell)}(f;x) = f^{(\ell)}(x) + \sum_{k=1}^{q} c_k^{[\ell]}(f;x)\, n^{-k} + \mathrm{o}\left(n^{-q}\right) \qquad (n \to \infty), \tag{9}$$

where the coefficients $c_k^{[\ell]}(f;x)$ are given by

$$c_k^{[\ell]}(f;x) = \left(\frac{d}{dx}\right)^{\ell} c_k(f;x) \tag{10}$$

and $c_k(f;x)$ is as defined in Theorem 1. Furthermore, the coefficients in (9) *have the representation*

$$\begin{aligned} c_k^{[\ell]}(f;x) &= \frac{1}{k!}\sum_{s=0}^{k} \frac{x^s}{s!} f^{(\ell+k+s)}(x) \sum_{m=0}^{k} \binom{k}{m} \frac{g^{(m)}(1)}{g(1)} \\ &\quad \times \sum_{r=0}^{s} (-1)^{s-r} \frac{\binom{s}{r}}{\binom{r+k+\ell}{r+m+\ell}} \sigma(r+k+\ell, r+m+\ell)\ . \end{aligned} \tag{11}$$

Remark 1 . *If $f \in K^{[\infty]}(x) = \bigcap_{q=1}^{\infty} K^{[q]}(x)$, the Jakimovski–Leviatan operators possess the complete asymptotic expansion*

$$P_n(f;x) \sim f(x) + \sum_{k=1}^{\infty} c_k(f;x)\, n^{-k} \qquad (n \to \infty),$$

where the coefficients $c_k(f;x)$ are as defined in (6) *and* (7) .

For the convenience of the reader we list explicit expressions of the initial coefficients $c_k(f;x)$:

$$\begin{aligned}
c_0(f;x) &= f(x) \\
c_1(f;x) &= \frac{1}{2} x f^{(2)}(x) + \frac{g'(1)}{g(1)} f'(x) \\
c_2(f;x) &= \frac{1}{8} x^2 f^{(4)}(x) + \frac{1}{6} x f^{(3)}(x) \left(1 + 3\frac{g'(1)}{g(1)}\right) \\
&\quad + \frac{1}{2} f^{(2)}(x) \frac{g'(1) + g^{(2)}(1)}{g(1)} \\
c_3(f;x) &= \frac{1}{48} x^3 f^{(6)}(x) + \frac{1}{24} x^2 f^{(5)}(x) \left(2 + 3\frac{g'(1)}{g(1)}\right) \\
&\quad + \frac{1}{24} x f^{(4)}(x) \left(1 + \frac{10g'(1) + 6g^{(2)}(1)}{g(1)}\right) \\
&\quad + \frac{1}{6} f^{(3)}(x) \frac{g'(1) + 3g^{(2)}(1) + g^{(3)}(1)}{g(1)}
\end{aligned}$$

In the special case of the Szász–Mirakjan operators we get the following corollary.

Corollary 3 (Complete asymptotic expansion for the derivatives of the Szász–Mirakjan operators S_n). *Let $q \in \mathbb{N}$, $x \geq 0$, $\ell \in \mathbb{N}_0$ and $f \in K^{[2q+2\ell]}(x)$. The derivatives of the Szász–Mirakjan operators satisfy the asymptotic relation*

$$S_n^{(\ell)}(f;x) = f^{(\ell)}(x) + \sum_{k=1}^{q} d_k^{[\ell]}(f;x)\, n^{-k} + \mathrm{o}\left(n^{-q}\right) \qquad (n \to \infty),$$

where the coefficients $d_k^{[\ell]}(f;x)$ are given by

$$d_k^{[\ell]}(f;x) = \frac{1}{k!} \sum_{s=0}^{k} \frac{x^s}{s!} f^{(\ell+k+s)}(x) \sum_{r=0}^{s} (-1)^{s-r} \frac{\binom{s}{r}}{\binom{r+k+\ell}{r+\ell}} \sigma(r+k+\ell, r+\ell) \,. \tag{12}$$

For $q = 2$, we obtain, for $f \in K^{[2\ell+4]}(x)$, the asymptotic relation

$$\begin{aligned}
& S_n^{(\ell)}(f;x) \\
= \; & f^{(\ell)}(x) + \frac{1}{2n}\left(x f^{(\ell+2)}(x) + \ell f^{(\ell+1)}(x)\right) \\
& + \frac{1}{48n^2}\left(6x^2 f^{(\ell+4)} + 4(3\ell+2) x f^{(\ell+3)} + 2\ell(3\ell+1) f^{(\ell+2)}\right) + o(n^{-2})
\end{aligned}$$

as $n \to \infty$. In the special case $q = 1$ Theorem 2 reveals the following Voronovskaja–type result.

Corollary 4 (Voronovskaja–theorem for the operators P_n). *For $x \geq 0$, $\ell \in \mathbb{N}_0$ and $f \in K^{[2\ell+2]}(x)$, the Jakimovski–Leviatan operators satisfy the asymptotic relation*

$$\begin{aligned} & \lim_{n\to\infty} n\left(P_n^{(\ell)}(f;x) - f^{(\ell)}(x)\right) \\ = \; & \frac{1}{2}\left(xf^{(\ell+2)}(x) + \ell f^{(\ell+1)}(x)\right) + \frac{g'(1)}{g(1)} f^{(\ell+1)}(x). \end{aligned}$$

It is not known to the authors if the specialization of Corollary 4 to the Szasz–Mirakjan operators

$$\lim_{n\to\infty} n\left(S_n^{(\ell)}(f;x) - f^{(\ell)}(x)\right) = \frac{1}{2}\left(xf^{(\ell+2)}(x) + \ell f^{(\ell+1)}(x)\right)$$

appears in the literature. The special case $\ell = 0$ of the latter formula is well–known.

The Kantorovich variant (4) is intimately connected to the operators (2) by the relation $K_n(f;x) = P_n'(F;x)$, where

$$F(x) = \int_0^x f(t)\; dt. \tag{13}$$

Therefore, we get as an immediate consequence of Theorem 2 the following corollary.

Corollary 5 (Complete asymptotic expansion for the operators K_n and their derivatives). *Let $q \in \mathbb{N}$, $x \geq 0$, $\ell \in \mathbb{N}_0$ and $f \in K^{[2q+2\ell+1]}(x)$. The operators K_n satisfy the asymptotic relation*

$$K_n^{(\ell)}(f;x) = f^{(\ell)}(x) + \sum_{k=1}^{q} c_k^{[\ell+1]}(F;x)\, n^{-k} + \mathrm{o}\left(n^{-q}\right) \qquad (n \to \infty),$$

where the coefficients $c_k^{[\ell]}(f;x)$ are as defined in Theorem 2 and the function F is given by Eq. (13).

Note that the assumption $f \in K^{[2q+2\ell+1]}(x)$ implies that $F'(x) = f(x)$. In the case $q = 1$ we obtain, for $f \in K^{[2\ell+3]}(x)$, the Voronovskaja–type result

$$\begin{aligned} & \lim_{n\to\infty} n\left(K_n^{(\ell)}(f;x) - f^{(\ell)}(x)\right) \\ = \; & \frac{1}{2}\left(xf^{(\ell+2)}(x) + (\ell+1) f^{(\ell+1)}(x)\right) + \frac{g'(1)}{g(1)} f^{(\ell+1)}(x). \end{aligned}$$

The result in [11, Theorem 3.1] would include the special case $\ell = 0$ of the latter formula if there had been shown that $\rho_n = o\left(n^{-1}\right)$ as $n \to \infty$. For

$f \in K^{[2\ell+5]}(x)$, we obtain

$$\begin{aligned} & K_n^{(\ell)}(f;x) \\ = \; & f^{(\ell)}(x) + \frac{1}{2n}\left(x f^{(\ell+2)}(x) + (\ell+1) f^{(\ell+1)}(x) + \frac{2g'(1)}{g(1)} f^{(\ell+1)}(x)\right) \\ & + \frac{1}{48n^2}\left(6x^2 f^{(\ell+4)}(x) + \left(4(3\ell+5)x + 24\frac{g'(1)}{g(1)}\right) f^{(\ell+3)}(x)\right. \\ & \left. + \left(2(\ell+1)(3\ell+4) + 24\frac{(\ell+2)g'(1) + g^{(2)}(1)}{g(1)}\right) f^{(\ell+2)}(x)\right) + o(n^{-2}) \end{aligned}$$

as $n \to \infty$.

3 Auxiliary results

It is easy to see that the derivatives of the Jakimovski–Leviatan operators possess the representation

$$P_n^{(\ell)}(f;x) = \frac{n^\ell e^{-nx}}{g(1)} \sum_{\nu=0}^{\infty} p_\nu(nx)\ \Delta_{1/n}^{\ell} f\left(\frac{\nu}{n}\right) \qquad (\ell = 0, 1, \ldots)$$

which was already remarked in [16, proof of Theorem 3]. We define, for $\lambda \geq 0$, the positive linear operators

$$P_{n,\lambda}(f;x) = \frac{e^{-nx}}{g(1)} \sum_{\nu=0}^{\infty} p_\nu(nx)\ f\left(\frac{\nu+\lambda}{n}\right)$$

in order to obtain the relation

$$P_n^{(\ell)}(f;x) = n^\ell \sum_{\lambda=0}^{\ell} (-1)^{\ell-\lambda} \binom{\ell}{\lambda} P_{n,\lambda}(f;x). \tag{14}$$

The essential result for the proof of the main theorem is the complete asymptotic expansion of the operators $P_{n,\lambda}$.

Proposition 6 (Complete asymptotic expansion for the operators $P_{n,\lambda}$). *Let $q \in \mathbb{N}$, $x \geq 0$, and $f \in K^{[2q]}(x)$. The operators $P_{n,\lambda}$ $(\lambda = 0, 1, \ldots)$ satisfy the asymptotic relation*

$$\begin{aligned} & P_{n,\lambda}(f;x) \\ = \; & \frac{1}{g(1)} \sum_{k=0}^{q} n^{-k} \sum_{s=0}^{k} \frac{x^s}{s!} f^{(k+s)}(x) \sum_{m=0}^{k} \frac{g^{(m)}(1)}{m!} b(\lambda, s, k, m) + \mathrm{o}\left(n^{-q}\right) \end{aligned}$$

as $n \to \infty$, where

$$b(\lambda, s, k, m) = \sum_{\mu=0}^{k-m} \binom{\lambda}{k-m-\mu} \sum_{r=0}^{s} (-1)^{s-r} \binom{s}{r} \frac{\sigma(r+k, r+k-\mu)}{(r+k)^{\underline{\mu}}}. \tag{15}$$

As a first step we calculate their moments. For each $r = 0, 1, \ldots$, we put $e_r(x) = x^r$.

Lemma 7 . *For all $r = 0, 1, \ldots$, the moments of the operators $P_{n,\lambda}$ $(\lambda = 0, 1, \ldots)$ possess the representation*

$$\begin{aligned} P_{n,\lambda}(e_r; x) &= \frac{1}{g(1)} \sum_{k=0}^{r} \frac{x^{r-k}}{n^k} \sum_{m=0}^{k} \frac{g^{(m)}(1)}{m!} \\ &\quad \times \sum_{\mu=0}^{k-m} \binom{\lambda}{k-m-\mu} \sigma(r, r-\mu) (r-\mu)^{\underline{k-\mu}} . \end{aligned}$$

Lemma 8 . *For all $s = 0, 1, \ldots$, the central moments of the operators $P_{n,\lambda}$ $(\lambda = 0, 1, \ldots)$ possess the representation*

$$P_{n,\lambda}\left((\cdot - x)^s ; x\right) = \frac{1}{g(1)} \sum_{k=0}^{s} \frac{s^{\underline{k}} x^{s-k}}{n^k} \sum_{m=0}^{k} \frac{g^{(m)}(1)}{m!} b(\lambda, s-k, k, m) \quad , \tag{16}$$

where the coefficients $b(\lambda, s, k, m)$ are as defined in Eq. (15) .

In order to show Proposition 6 we apply a general approximation theorem for positive linear operators due to Sikkema [20, Theorems 1 and 2].

Lemma 9 . *Let $q \in \mathbb{N}$ and $x \geq 0$. Moreover, let $L_n : K^{[2q]}(x) \to C[0, \infty)$ be a sequence of positive linear operators. If, for $s = 0, 1, \ldots, 2q + 2$,*

$$L_n\left((\cdot - x)^s ; x\right) = \mathcal{O}\left(n^{-\lfloor (s+1)/2 \rfloor}\right) \qquad (n \to \infty) , \tag{17}$$

then we have, for each bounded function $f \in K^{[2q]}(x)$,

$$L_n(f; x) = \sum_{s=0}^{2q} \frac{1}{s!} f^{(s)}(x)\, L_n\left((\cdot - x)^s ; x\right) + \mathrm{o}\left(n^{-q}\right) \qquad (n \to \infty) .$$

In order to apply Lemma 9, we have to check whether the operators $P_{n,\lambda}$ satisfy condition (17) .

Lemma 10 . *For each $x \geq 0$ and all $s = 0, 1, \ldots$, the central moments of the operators $P_{n,\lambda}$ $(\lambda = 0, 1, \ldots)$ satisfy the estimation*

$$P_{n,\lambda}\left((\cdot - x)^s ; x\right) = \mathcal{O}\left(n^{-\lfloor (s+1)/2 \rfloor}\right) \qquad (n \to \infty) .$$

Since Lemma 9 applies only to bounded functions f we shall need the following lemma for the proof of Proposition 6.

Lemma 11 . *Let $f \in E$, $\lambda \geq 0$, $x \geq 0$, and $d > 0$. Assume that $f(t) = 0$, for all $t \in (x - d, x + d) \cap [0, \infty)$. Then there exists a constant $c > 0$, independent of n, such that*

$$P_{n,\lambda}(f; x) = \mathcal{O}\left(e^{-cn}\right) \qquad (n \to \infty) .$$

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